Bifurcation Buckling of Circular Cylindrical Shells Under Uniform External Pressure

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This paper presents asymptotic solutions for the eigenvalue problems of buckling under uniform external pressure of a circular cylindrical shell having an arbitrary combination of the boundary conditions for the simply supported, clamped, and free ends. A simple formula for the buckling pressure is derived, which is shown to be accurate enough for engineering purposes by comparison with available results. The eigenvalues calculated for all possible combinations of the boundary conditions show that the buckling pressures are affected significantly by the presence of a free end as well as axial constraint at a supported end.

I. Introduction

THE eigenvalue problems of the buckling of circular cylindrical shells under uniform external pressure have been investigated by many researchers. Analytical as well as numerical solutions have been obtained most extensively for shells simply supported in the classical sense or rigidly clamped at both ends. In cases of shells with simply supported ends, analytical solutions can be obtained in closed form following a procedure suggested by Flügge,[1] and some practical formulas for the buckling pressure are available, including those proposed by von Mises,[2] Batdorf,[3] and Armenàkas and Herrmann.[4] In cases of shells with clamped ends, the buckling pressures have been calculated approximately by Nash[5] and by Galletly and Bart[6] using the Rayleigh-Ritz or the Galerkin method. The results show that the buckling pressures of the clamped shells are much higher than those of the simply supported shells. The results obtained by Singer[7] for a shell with axial elastic springs that permitted no rotation at the ends indicate that the increase in the buckling pressure of the clamped shells is attributed to the axial constraint rather than the rotational constraint. The significance of the effect of the axial constraint gained support from the numerical solutions obtained by Sobel.[8] There are eight different sets of boundary conditions that we can impose at a supported end: four variations for each of the simply supported and clamped ends, depending on the in-plane boundary conditions. Sobel calculated the buckling pressures in eight cases, imposing the same sets of the boundary conditions at both ends. The results show that there is a significant difference between the buckling pressures in the cases with and without axial constraint regardless of rotational constraint. The values of the buckling pressure are hardly distinguishable within each group when the length-to-radius ratio is greater than two. A similar trend is observed in the natural frequencies calculated by Forsberg[9] for the free vibrations of a circular cylindrical shell. This indicates that the modal characteristics are similar for both buckling under uniform pressure and free vibrations. In the buckling analyses, two types of pressure loading have been considered: lateral pressure loading and hydrostatic pressure loading. It is well accepted that, if the shell has moderate to great length, the effects of axial stress due to hydrostatic pressure loading are negligible, and buckling pressures are not much different between the two loading systems.

In the present paper, asymptotic solutions are obtained for the eigenvalue problems, considering all possible combinations of the sets of the boundary conditions for simply supported, clamped, and free ends. Nine different sets of boundary conditions (eight for the simply supported and the clamped end and one for the free end) make 45 combinations between two ends. The characteristic equations are derived for all 45 combinations. It turns out that only five different types of them exist, depending on whether the end is free or supported and whether or not the supported end is axially constrained. A simple formula is derived, from which the buckling pressure can easily be calculated for any of the 45 combinations of the sets of boundary conditions. The buckling pressures calculated with the aid of this formula are compared with those available in the literature. The axial stress due to the hydrostatic pressure loading is taken into account in the formulation of the governing equations, but it will be shown that it has no effect on the asymptotic solutions. The analysis follows the procedure developed in the first author's previous papers[10,11] dealing with free vibrations. The reader is referred to the previous papers for details of the mathematical development.

II. Fundamental Assumptions and Governing Equations

An elastic circular cylindrical shell having a radius R, a thickness h, and a length $2L$ is compressed uniformly under external pressure p. The uniformly stressed state characterized by the axial and circumferential stress resultants N_{x0} and $N_{\theta 0}$ is slightly perturbed to represent a bifurcation of the equilibrium state corresponding to the buckling. Two types of pressure loading will be considered: lateral and hydrostatic pressure loading, under which the initial stress resultants N_{x0} and $N_{\theta 0}$ are given by

$$N_{\theta 0} = -pR, \qquad N_{x0} \begin{cases} = 0 \text{ for lateral pressure} \\ = -pR/2 \text{ for hydrostatic pressure} \end{cases} \tag{1}$$

where p is positive for external pressure.

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The governing equations formulated by Budiansky[12] for such small perturbations of stressed shells are specialized for the pressurized circular cylindrical shell, taking the x and θ coordinates along the axis and the circumference. These equations are written in a nondimensional form by introducing nondimensional quantities and operators defined as

$$\ell = \frac{L}{R}, \qquad \delta = \frac{h^2}{12R^2}, \qquad 4k^4 = 12(1-\nu^2)\left(\frac{R}{h}\right)^2$$

$$y = \frac{x}{R}, \qquad u = \frac{u_x}{R}, \qquad v = \frac{u_\theta}{R}, \qquad w = \frac{w_z}{R}$$

$$N = \frac{N_x}{K}, \qquad M = \frac{RM_x}{D}, \qquad Q = \frac{Q_x}{K}, \qquad S = \frac{S_{x\theta}}{K}$$

$$q = \frac{N_{\theta 0}}{K}, \qquad \sigma = \frac{N_{x0}}{K}$$

$$(\ )' = \frac{\partial(\ )}{\partial y}, \quad (\ )^{\cdot} = \frac{\partial(\ )}{\partial \theta}, \quad \nabla^2(\ ) = (\ )'' + (\ )^{\cdot\cdot} \tag{2}$$

where u_x, u_θ, and w_z are the axial, circumferential, and lateral displacements, N_x and M_x the axial stress resultant and couple, and Q_x and $S_{x\theta}$ the lateral and circumferential components of the equivalent edge shear given by

$$Q_x = \frac{\partial M_x}{\partial x} + \left(\frac{2}{R}\right)\frac{\partial M_{x\theta}}{\partial \theta} + N_{x0}\frac{\partial w_z}{\partial x}$$

$$S_{x\theta} = N_{x\theta} + \left(\frac{3}{2R}\right)M_{x\theta} + N_{x\theta}\frac{\partial u_\theta}{\partial x} \tag{3}$$

$$K = \frac{Eh}{1-\nu^2}, \qquad D = \frac{Eh^3}{12(1-\nu^2)} \tag{4}$$

with Young's modulus E and Poisson's ratio ν.

The equilibrium equations are written in terms of the displacements, which are then reduced to a single equation for the deflection function w. If small terms are neglected, the equation becomes

$$\nabla^8 w + 8w''^{\cdot\cdot\cdot\cdot} + 2w^{\cdot\cdot\cdot\cdot\cdot\cdot} + (1-\nu^2)w''''/\delta + 4w''^{\cdot\cdot} + w^{\cdot\cdot\cdot\cdot}$$
$$-(q/\delta)[\nabla^4 w^{\cdot\cdot} + 3w''^{\cdot\cdot} + w^{\cdot\cdot\cdot\cdot}]$$
$$-(\sigma/\delta)[\nabla^4 w'' - w''^{\cdot\cdot}] = 0 \tag{5}$$

Also, those quantities to be prescribed as boundary conditions are expressed only in terms of w to obtain the following relations:

$$\nabla^4 u = -\nu w''' + w'^{\cdot\cdot}$$
$$\nabla^4 v = -(2+\nu)w''^{\cdot} - w^{\cdot\cdot\cdot}$$
$$\nabla^4 N = (1-\nu^2)w''^{\cdot\cdot} + \delta\nu(w^{\cdot\cdot\cdot\cdot\cdot\cdot} + w^{\cdot\cdot\cdot\cdot}) + (q+\nu\sigma)w''''$$
$$\nabla^4 M = -\nabla^4(w'' + \nu w^{\cdot\cdot}) - \nu(2+\nu)w''^{\cdot\cdot} - \nu w^{\cdot\cdot\cdot\cdot}$$
$$\nabla^4 Q = -\nabla^4[w''' + (2-\nu)w'^{\cdot\cdot}] - 3w'''^{\cdot\cdot}$$
$$-(2-\nu)w'^{\cdot\cdot\cdot\cdot} + (\sigma/\delta)\nabla^4 w'$$
$$\nabla^4 S = -(1-\nu^2)w'''^{\cdot} - \delta(2-\nu)(w'^{\cdot\cdot\cdot\cdot\cdot} + w'^{\cdot\cdot\cdot}) - \sigma w'^{\cdot\cdot\cdot} \tag{6}$$

It has been assumed in deriving Eqs. (5) and (6) that the shell is thin:

$$h/R \ll 1 \tag{7a}$$

so that

$$\delta \ll 1 \tag{7b}$$

It has also been assumed that the values of the initial stress are much smaller than the values of Young's modulus, so that

$$q \ll 1, \qquad \sigma \ll 1 \tag{8}$$

A further simplification has been achieved by assuming that

$$\delta w'' \ll w, \qquad \delta w^{\cdot\cdot} \ll w \tag{9}$$

Let the fundamental solutions of Eq. (5) be written in the form

$$w = W_0 \exp(\lambda y)\cos n\theta \tag{10}$$

where W_0 is an arbitrary constant, n the circumferential wave number, and λ an eigenvalue that determines the modal characteristics along the generator. Then, it follows from Eqs. (9) that

$$\delta|\lambda^2| \ll 1, \qquad \delta n^2 \ll 1 \tag{11}$$

This is equivalent to stating that the wavelength is much greater than the wall thickness.

Boundary conditions to be considered in the present paper are those of the simply supported, clamped, and free ends. They are defined and designated as follows:

Simply supported ends:

$$S1(w = M = u = v = 0), \quad S2(w = M = u = S = 0)$$
$$S3(w = M = N = v = 0), \quad S4(w = M = N = S = 0) \tag{12a}$$

Clamped ends:

$$C1(w = w' = u = v = 0), \quad C2(w = w' = u = S = 0)$$
$$C3(w = w' = N = v = 0), \quad C4(w = w' = N = S = 0) \tag{12b}$$

Free ends:

$$FR(Q = M = N = S = 0) \tag{12c}$$

III. Characteristic Equations and Eigenvalues

Substituting w from Eq. (10) into Eq. (5), we obtain

$$\lambda^8 + A_3\lambda^6 + A_2\lambda^4 + A_1\lambda^2 + A_0 = 0 \tag{13}$$

where

$$A_0 = n^4(n^2-1)[n^2 - 1 + q/\delta]$$
$$A_1 = -4n^2(n^2-1)^2 - n^2(n^2+1)\sigma/\delta - n^2(2n^2-3)q/\delta$$
$$A_2 = (1-\nu^2)/\delta, \qquad A_3 = -4n^2 - \sigma/\delta \tag{14}$$

We shall solve Eq. (13) beginning with the case of $A_0 = 0$ and then proceeding to the case of $A_0 \neq 0$.

In the case of $A_0 = 0$, the first of Eqs. (14) yields a value of q which corresponds to the buckling pressure of a ring and will be denoted here as q_0:

$$q_0 = -\delta(n^2-1) \tag{15}$$

The solutions of Eq. (13) are given in the form

$$\lambda_1, \lambda_2 = 0, \qquad \lambda_3, \lambda_4 = \pm n\xi_1$$
$$\lambda_5, \lambda_6, \lambda_7, \lambda_8 = \pm n(\xi_2 \pm i\eta_2) \tag{16}$$

where ξ_1, ξ_2, and η_2 are positive real, and $i = (-1)^{1/2}$.

The root and coefficient relations of Eq. (13) read

$$\xi_1^2 + 2(\xi_2^2 - \eta_2^2) = 4 + \sigma_0/\delta n^2$$

$$2\xi_1^2(\xi_2^2 - \eta_2^2) + (\xi_2^2 - \eta_2^2)^2 = (1 - \nu^2)/\delta n^4$$

$$\xi_1^2(\xi_2^2 - \eta_2^2)^2 = (n^2 - 1)(2n^2 - 1)/n^4 + (n^2 + 1)\sigma_0/\delta n^4 \quad (17)$$

where

$$\sigma_0 \begin{cases} = 0 \text{ for lateral pressure} \\ = q_0/2 \text{ for hydrostatic pressure} \end{cases}$$

Let a geometric parameter Δ be defined by

$$\Delta = n^2/2k^2 = \delta^{1/2} n^2/(1 - \nu^2)^{1/2} \quad (18)$$

and assumed much smaller than unity:

$$\Delta \ll 1 \quad (19)$$

Then, approximate solutions of Eqs. (17) are obtained by expanding ξ_1^2, ξ_2^2, and η_2^2 into asymptotic series in Δ^2 and Δ such that

$$\xi_1^2 = \Delta^2(\xi_{10} + \Delta^2\xi_{11} + \Delta^4\xi_{12} + \cdots)$$

$$\xi_2^2 = \Delta^{-1}(\xi_{20} + \Delta\xi_{21} + \Delta^2\xi_{22} + \cdots)$$

$$\eta_2^2 = \Delta^{-1}(\eta_{20} + \Delta\eta_{21} + \Delta^2\eta_{22} + \cdots) \quad (20)$$

where ξ_{ij} and η_{ij} are real numbers of order of magnitude unity.

Substituting Eqs. (20) in Eqs. (17) and comparing terms of like powers of Δ between the left- and right-hand members, we obtain

$$\xi_1 = \Delta\xi_0 + O(\Delta^2)$$

$$\xi_2 = \eta_2 = (1/2\Delta)^{1/2} + O(\Delta) \quad (21)$$

with

$$\xi_0 = (\xi_{10})^{1/2} = [(n^2 - 1)(2n^2 - 1)/n^4 + (n^2 + 1)\sigma_0/\delta n^4]^{1/2} \quad (22)$$

Thus, neglecting small terms of order of magnitude Δ, we have the first approximation solutions for λ such that

$$\lambda_1, \lambda_2 = 0, \qquad \lambda_3, \lambda_4 = \pm\Delta n\xi_0$$

$$\lambda_5, \lambda_6, \lambda_7, \lambda_8 = \pm(1 \pm i)n/(2\Delta)^{1/2} \quad (23)$$

Let it be assumed that

$$1/n\ell = O(\Delta^{1/2}) \quad (24)$$

Then, the general solution for w can be written in the form

$$w = \left[W_1 + W_2 y + W_3 y^2 + W_4 y^3 + \sum_{j=5}^{8} w_j \exp(\lambda_j y) \right] \cos n\theta \quad (25)$$

where W_m ($m = 1,2,\ldots,8$) are arbitrary constants to be determined from consideration of the boundary conditions.

The homogeneous equations constituting the boundary conditions are expressed in terms of W_m with the aid of Eqs. (25) and (6). The results are identical to those derived in the previous paper[10] dealing with the inextensional free vibrations. The conclusions concerning nontrivial solutions for the inextensional free vibrations therefore hold for the buckling. The nontrivial solutions for the buckling deflection function w are as follows:

For *FR-FR*,

$$w = W_0 \cos n\theta \quad \text{(Rayleigh mode)}$$

$$w = W_0(y/\ell)\cos n\theta \quad \text{(Love mode)} \quad (26)$$

For *FR-SF*,

$$w = W_0(1 + y/\ell)\cos n\theta \quad (27)$$

where W_0 is an arbitrary constant, and *SF* represents *S*3, *S*4, *C*3, and *C*4, which are characterized by the conditions $w = N = 0$.

In the case in which $A_0 \neq 0$, the solutions of Eq. (13) take the form

$$\lambda_1, \lambda_2 = \pm n\xi_1, \qquad \lambda_3, \lambda_4 = \pm in\eta_1$$

$$\lambda_5, \lambda_6, \lambda_7, \lambda_8 = \pm n(\xi_2 \pm i\eta_2) \quad (28)$$

if q and σ are assumed to be of the same order of magnitude as q_0, so that

$$q/\delta n^2 = O(1), \qquad \sigma/\delta n^2 = O(1) \quad (29)$$

The root and coefficient relations of Eq. (13) read

$$2(\xi_2^2 - \eta_2^2) + (\xi_1^2 - \eta_1^2) = -A_3/n^2$$

$$(\xi_2^2 + \eta_2^2)^2 - \xi_1^2\eta_1^2 + 2(\xi_2^2 - \eta_2^2)(\xi_1^2 - \eta_1^2) = A_2/n^4$$

$$2\xi_1^2\eta_1^2(\xi_2^2 - \eta_2^2) - (\xi_1^2 - \eta_1^2)(\xi_2^2 + \eta_2^2)^2 = A_1/n^6$$

$$\xi_1^2\eta_1^2(\xi_2^2 + \eta_2^2)^2 = -A_0/n^8 \quad (30)$$

Let ξ_1^2, η_1^2, ξ_2^2, and η_1^2 be expanded into asymptotic series in Δ such that

$$\xi_1^2 = \Delta(\xi_{10} + \Delta\xi_{11} + \Delta^2\xi_{12} + \cdots)$$

$$\eta_1^2 = \Delta(\eta_{10} + \Delta\eta_{11} + \Delta^2\eta_{12} + \cdots)$$

$$\xi_2^2 = \Delta^{-1}(\xi_{20} + \Delta\xi_{21} + \Delta^2\xi_{22} + \cdots)$$

$$\eta_2^2 = \Delta^{-1}(\eta_{20} + \Delta\eta_{21} + \Delta^2\eta_{22} + \cdots) \quad (31)$$

where ξ_{ij} and η_{ij} are real numbers of order of magnitude unity. Proceeding as before, we obtain from the first three of Eqs. (30)

$$\xi_1 = \eta_1 = \Delta^{1/2}\xi_{10}^{1/2} + O(\Delta)$$

$$\xi_2 = \eta_2 = 1/(2\Delta)^{1/2} + O(\Delta) \quad (32)$$

where ξ_{10} is as yet unknown, to be determined from consideration of the boundary conditions. Thus, we have the first approximation solutions for λ as

$$\lambda_1, \lambda_2 = \pm n\xi_1, \qquad \lambda_3, \lambda_4 = \pm in\xi_1$$

$$\lambda_5, \lambda_6, \lambda_7, \lambda_8 = \pm n(1 \pm i)/(2\Delta)^{1/2} \quad (33)$$

Here, ξ_1 is used as the unknown instead of ξ_{10}, on the understanding that it is of order of magnitude $\Delta^{1/2}$.

The general solution for w is now given as

$$w = \left[\sum_{i=1}^{4} W_i \exp(\lambda_i y) + \sum_{j=5}^{8} W_j \exp(\lambda_j y) \right] \cos n\theta \quad (34)$$

where W_m ($m = 1,2,\ldots,8$) are arbitrary constants. The first group of terms with subscript i represents the global solutions, whereas the second with subscript j represents the edge-zone solutions.

The last of Eqs. (30) yields the following formula for the buckling pressure:

$$q = q_0[1 + (1 - \nu^2)\xi_1^4/\delta(n^2 - 1)^2] \tag{35}$$

It should be noted here that Eq. (35) becomes identical to the formula derived in the previous paper for the natural frequencies if q and q_0 are replaced by the frequency parameters ω^2 and ω_0^2, respectively.

The homogeneous equations constituting the boundary conditions are expressed in terms of W_m with the aid of Eqs. (34) and (6). It turns out that the resulting expressions are identical to those derived in the author's previous paper.[11] The conclusions of the previous paper concerning the characteristic equations for the free vibrations therefore hold for the buckling as well. Consequently, if the inextensional buckling is dealt with inclusively, five types of the characteristic equations result, depending on the combination of the representative boundary conditions; *SR, SF,* and *FR*. Here, *SR* represents *S*1, *S*2, *C*1, and *C*2, which are characterized by the conditions $w = u = 0$:

Type I (*SR-SR*):

$$\cosh 2n\xi_1\ell \cos 2n\xi_1\ell - 1 = 0 \tag{36a}$$

Type II (*SR-SF*):

$$\cosh 2n\xi_1\ell \sin 2n\xi_1\ell - \sinh 2n\xi_1\ell \cos 2n\xi_1\ell = 0 \tag{36b}$$

Type III (*SF-SF*):

$$\sin 2n\xi_1\ell = 0 \tag{36c}$$

Type IV (*FR-SR*):

$$\cosh 2n\xi_1\ell \cos 2n\xi_1\ell + 1 = 0 \tag{36d}$$

Type V (*FR-FR* and *FR-SF*):

$$\xi_1^2 = 0 \tag{36e}$$

The characteristic equations and the combinations of the boundary conditions are summarized in Table 1.

In the buckling, unlike in the free vibrations, only the minimum of the characteristic roots are needed for the calculation of the buckling pressure. These are given by

$$2n\xi_1\ell \begin{cases} = 4.730 \text{ (type I)}, \quad = 3.927 \text{ (type II)}, \\ = \pi \text{ (type III)}, \quad = 1.875 \text{ (type IV)}, \\ = 0.0 \text{ (type V)} \end{cases} \tag{37}$$

The buckling pressure can now be calculated with the aid of Eq. (35) using the values of ξ_1 as specified in Eqs. (37).

It is seen as in the free vibrations that the buckling characteristics depend on whether an end is free or supported, and whether or not the supported end is allowed to move freely in the axial direction. It should be noted that σ is not involved in Eq. (35) or (36). This indicates that there is no significant difference in the buckling characteristics, regardless of whether the loading conditions are lateral or hydrostatic. It is also interesting to note that the use of Donnell's equations leads to an expression for the buckling pressure, which becomes identical in form to Eq. (35) when $n^2 \gg 1$. Equations (37) indicate that ξ_1 becomes vanishingly small as ℓ becomes large, so that the buckling mode becomes inextensional and the buckling pressure is given by q_0.

IV. Comparison with Other Solutions

Values of δ and ℓ are given when the shell geometry is known. The type of the characteristic equation is determined from the boundary conditions. Accordingly, a characteristic root is chosen from Eqs. (37), and the value of ξ_1 is calculated for a given integer value of n. Thus, the value of the pressure parameter q is calculated from Eq. (35). The calculation is repeated for various values of n. A critical value of q corresponding to the buckling pressure is determined as the minimum of q for all the values of n.

The buckling pressures are calculated from Eq. (35) for various values of δ and ℓ for type III. The result is presented in Fig. 1 and compared with more accurate solutions obtained by Flügge[1] for *S*3-*S*3 under the lateral pressure loading. Also shown in Fig. 1 is a result from Donnell's equations. There is no appreciable difference between the results from the present approximation and Flügge's solutions, whereas the result from Donnell's equations deviates substantially from the other two as ℓ increases and n decreases. A similar trend is observed in Table 2, in which values of q/δ and n for type III calculated from Eq. (35) are compared with those calculated

Table 1 Characteristic equations and combinations of boundary conditions

Type	Characteristic equations	Combinations of boundary conditions	Representative boundary conditions
I	$\cosh 2n\xi_1\ell \times \cos 2n\xi_1\ell - 1 = 0$	*S*1-*S*1, *S*1-*S*2, *S*1-*C*1, *S*1-*C*2 *S*2-*S*2, *S*2-*C*1, *S*2-*C*2, *C*1-*C*1 *C*1-*C*2, *C*2-*C*2	*SR-SR*
II	$\cosh 2n\xi_1\ell \sin 2n\xi_1\ell - \sinh 2n\xi_1\ell \cos 2n\xi_1\ell = 0$	*S*1-*S*3, *S*1-*S*4, *S*1-*C*3, *S*1-*C*4 *S*2-*S*3, *S*2-*S*4, *S*2-*C*3, *S*2-*C*4 *C*1-*S*3, *C*1-*S*4, *C*1-*C*3, *C*1-*C*4 *C*2-*S*3, *C*2-*S*4, *C*2-*C*3, *C*2-*C*4	*SR-SF*
III	$\sin 2n\xi_1\ell = 0$	*S*3-*S*3, *S*3-*S*4, *S*3-*C*3, *S*3-*C*4 *S*4-*S*4, *S*4-*C*3, *S*4-*C*4, *C*3-*C*3 *C*3-*C*4, *C*4-*C*4	*SF-SF*
IV	$\cosh 2n\xi_1\ell \cos 2n\xi_1\ell + 1 = 0$	*FR-S*1, *FR-S*2, *FR-C*1, *FR-C*2	*FR-SR*
V	$\xi_1^2 = 0$	*FR-FR*	*FR-FR*
		*FR-S*3, *FR-S*4, *FR-C*3, *FR-C*4	*FR-SF*

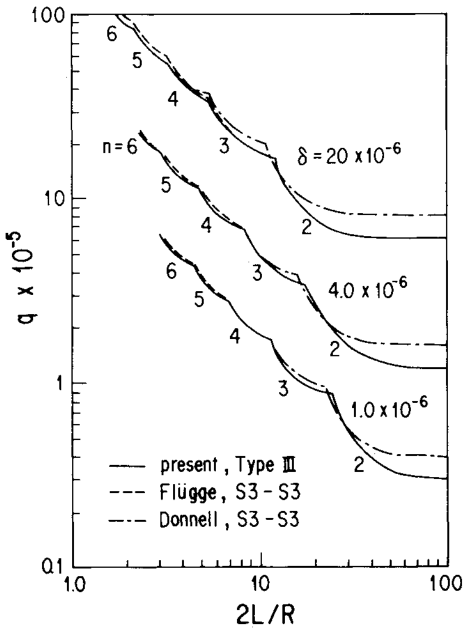

Fig. 1 Comparison with results from Flügge's and Donnell's equations (lateral pressure).

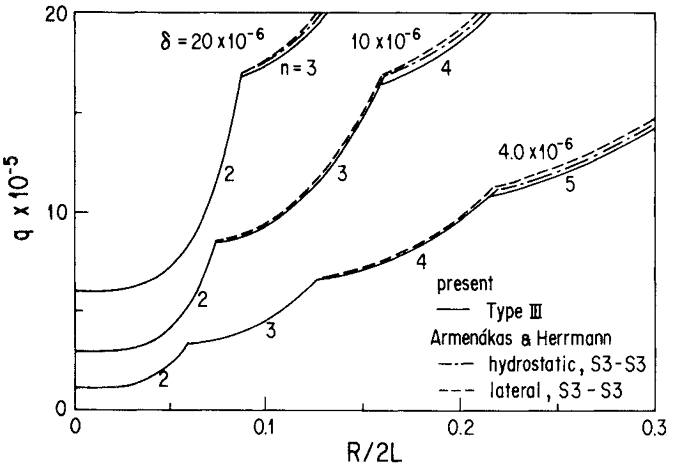

Fig. 2 Comparison with Armenàkas and Herrmann's calculations (hydrostatic pressure).

Table 2 Comparison of buckling pressures

		q/δ (n)		
			Simitses and Aswani ($S3$-$S3$)	
$\frac{2L}{\pi R}$	$\frac{R}{h}$	Present (type III)	Budiansky-Koiter eqs.	Donnell eqs.
1/3	35	57.4018 (7)	70.5175 (6)	69.2578 (6)
3		6.44059 (2)	6.44560 (2)	5.32740 (2)
15		3.00550 (2)	3.01306 (2)	3.21080 (2)
100		3.00000 (2)	3.00000 (2)	3.20000 (2)
1/3	200	134.738 (10)	147.483 (10)	146.813 (10)
3		16.4043 (3)	16.2343 (3)	14.7208 (3)
15		3.17975 (2)	3.18545 (2)	3.31345 (2)
100		3.00000 (2)	3.00000 (2)	3.20000 (2)
1/3	1000	301.999 (15)	313.523 (15)	312.853 (15)
3		32.9877 (5)	33.0159 (5)	32.3756 (5)
15		7.49383 (2)	7.40602 (2)	5.83350 (2)
100		3.00227 (2)	3.00000 (2)	3.20000 (2)

for $S3$-$S3$ under the lateral pressure loading by Simitses and Aswani[13] using Budiansky-Koiter's and Donnell's equations.

Approximate solutions have been derived in closed form by von Mises,[2] Batdorf,[3] and Armenàkas and Herrmann,[4] which have been used as practical formulas for calculating the buckling pressure. Without exception, they have been derived in the case of $S3$-$S3$. Armenàkas and Herrmann's formula reads in our notation,

$$q_{AH} = -q_0 \frac{1 + \xi_1^2 + (1 - \nu^2)\xi_1^4/\delta(n^2 - 1)^2(1 + \xi_1^2)^2}{1 + (\sigma/q)\xi_1^2} \tag{38}$$

with $\xi_1 = \pi/2n\ell$ for type III. When $n^2 \gg 1$, Eq. (38) reduces to the formula of von Mises and Batdorf. Under the assumption of Eq. (23), $1/n\ell = O(\Delta^{1/2})$, Armenàkas and Herrmann's formula becomes identical to Eq. (35). The buckling pressures calculated by Armenàkas and Herrmann are plotted in Fig. 2 and compared with those of the present approximation. A good agreement is observed between the two results. Armenàkas and Herrmann have shown that there is no significant effect of the axial stress due to the hydrostatic pressure loading. A similar observation has been made concerning the effect of the hydrostatic pressure loading by Soong,[14] who calculated the buckling pressures for $S3$-$S3$ using the Sanders

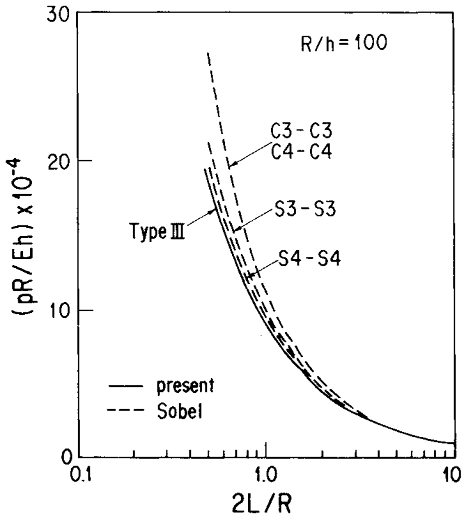

Fig. 3 Comparison with Sobel's numerical solutions (hydrostatic pressure).

Table 3 Comparison of buckling stress ratios

		$\sigma_\theta/\sigma_{cl}$[a]		
			Soong (Sanders' theory, $S3$-$S3$)	
$\frac{2L}{R}$	$\frac{R}{h}$	Present (type III)	Lateral	Hydrostatic
1.0	100	0.1517 (9)	0.1741 (8)	0.1614 (8)
4.0		0.0391 (4)	0.0397 (4)	0.0389 (4)
10.0		0.0146 (3)	0.0148 (3)	0.0147 (3)
40.0		0.00467 (2)	0.00468 (2)	0.00468 (2)
1.0	500	0.0676 (13)	0.0719 (13)	0.0699 (13)
4.0		0.0173 (7)	0.0176 (7)	0.0174 (7)
10.0		0.00663 (4)	0.00667 (4)	0.00664 (4)
40.0		0.00156 (2)	0.00156 (2)	0.00156 (2)
1.0	1500	0.0391 (17)	0.0405 (17)	0.0397 (17)
4.0		0.0099 (9)	0.0099 (9)	0.0099 (9)
10.0		0.00403 (5)	0.00404 (5)	0.00403 (5)
40.0		0.00095 (3)	0.00095 (3)	0.00095 (3)

[a]σ_θ: buckling hoop stress $= Eq/(1-\nu^2)$; σ_{cl}: classical buckling stress for axial compression $= E(h/R)[3(1-\nu^2)]^{-1/2}$.

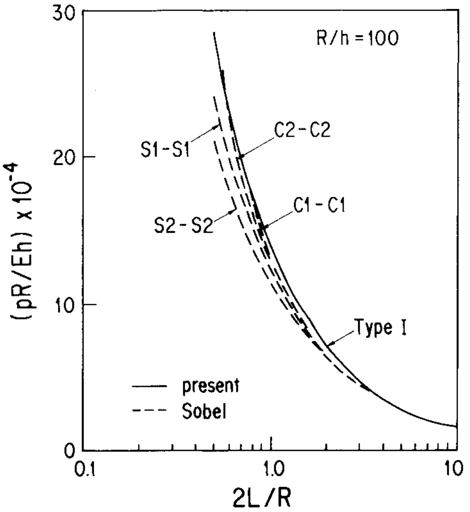

Fig. 4 Comparison with Sobel's numerical solutions (hydrostatic pressure).

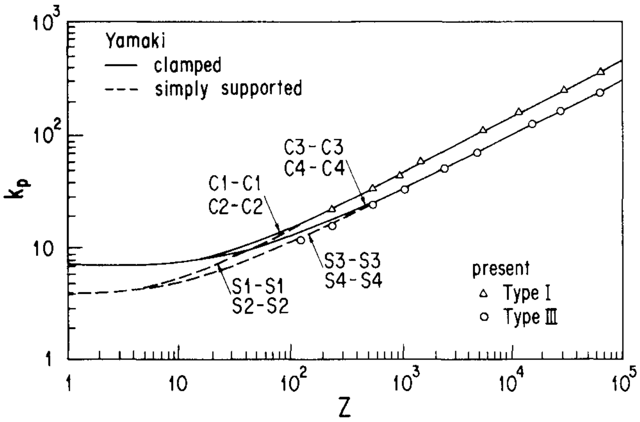

Fig. 5 Comparison with Yamaki's numerical solutions (lateral pressure); $k_p = pR(2L)^2/\pi^2 D$, $Z = (1-\nu^2)^{1/2}(2L)^2/Rh$.

theory. The accuracy of the present approximation is tested in Table 3 by comparison with Soong's results. A good agreement is observed again.

An extensive numerical calculation conducted by Sobel[8] for the buckling under the hydrostatic pressure loading has shown that the buckling pressures for $S1$-$S1$, $S2$-$S2$, $C1$-$C1$, and $C2$-$C2$ are much higher than those for $S3$-$S3$, $S4$-$S4$, $C3$-$C3$, and $C4$-$C4$, whereas there is no appreciable difference within each group of the combinations of the boundary conditions. The buckling pressures in the case of $R/h = 100$ calculated by Sobel for the first and the second group of the boundary conditions are presented in Figs. 3 and 4, together with those calculated from Eq. (35) using the eigenvalues of types I and III as given in Eqs. (37). The agreement of the curves is satisfactory in general, and it becomes excellent when $\ell \geq 2$. The buckling pressure under the lateral pressure loading has been calculated by Yamaki,[15] who imposed the same combinations of the boundary conditions as Sobel. The results are shown by the curves in Fig. 5. The results from Eq. (35) are also plotted in Fig. 5, and they agree so well with the points on the curves that one can hardly recognize the difference in the log-log scale. The experimental results of Weingarten and Seide[16] and Yamaki and Otomo[17] are plotted in Fig. 6, together with the theoretical curves calculated by Yamaki and Otomo for $C1$-$C1$ and $S3$-$S3$ and those calculated with the aid of Eq. (35) for types I, II, and III. The test specimens were bonded at both ends to thick plates. One of the endplates was

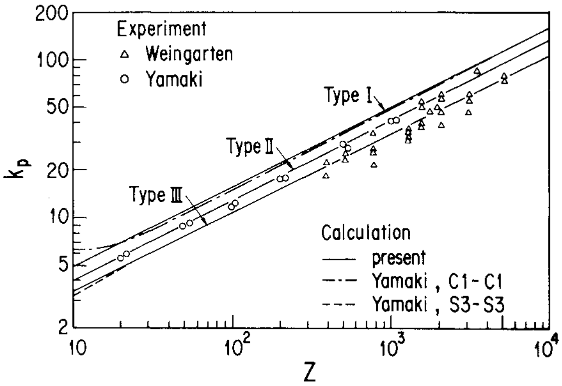

Fig. 6 Comparison with experiments by Weingarten and Seide and by Yamaki and Otomo (hydrostatic pressure); $k_p = pR(2L)^2/\pi^2 D$, $Z = (1-\nu^2)^{1/2}(2L)^2/Rh$.

fixed to a baseplate of testing machines, whereas the other was left unrestrained relative to the baseplate. The pressure loading was hydrostatic. Weingarten and Seide used Batdorf's formula for $S3$-$S3$ to make a theoretical prediction. Yamaki and Otomo used their numerical solutions for $C1$-$C1$ as a basis of comparison and attributed the discrepancy between the theory and the experiment to initial imperfections. It is apparent, however, that the boundary conditions of the specimens are represented precisely by neither $S3$-$S3$ nor $C1$-$C1$. They are more likely to be characterized by something between SR-SF and SR-SR. It follows from the present analysis, therefore, that the buckling pressure should be estimated by a straight line located somewhere between types I and II. One now can see that the experimental results are much closer to the theoretical prediction.

V. Conclusions

It has been shown that five types of the characteristic equations exist, depending on the combinations of the representative boundary conditions: $SR(w = u = 0)$, $SF(w = N = 0)$, and $FR(N = S = 0)$. It may be stated, therefore, that the buckling characteristics depend on whether an end is free or supported, and whether or not the supported end is allowed to move freely in the axial direction. The buckling pressure can be calculated accurately enough for engineering purposes by the formula

$$q = -\delta(n^2-1)[1 + (1-\nu^2)\xi_1^4/\delta(n^2-1)^2]$$

where ξ_1 is given according to the combinations of the boundary conditions specified as follows:

$$\xi_1 \begin{cases} = 4.730/2n\ell\ (SR\text{-}SR), & = 3.927/2n\ell\ (SR\text{-}SF) \\ = 3.142/2n\ell\ (SF\text{-}SF), & = 1.875/2n\ell\ (SR\text{-}FR) \\ = 0.0\ (FR\text{-}FR \text{ and } FR\text{-}SF) \end{cases}$$

where ℓ is the half-length L divided by the radius R. The stress parameter σ does not appear in the preceding formulas. This implies that the axial stress due to hydrostatic pressure has no significant effect on the buckling pressure. The error estimation in the previous paper for the free vibrations is valid for the present solutions for the buckling, with errors of order of magnitude $\Delta^{1/2}$ and $1/n\ell$. To be more specific, one may expect, on the basis of the comparison with other solutions, that the preceding formulas are applicable in cases in which $Z = (1-\nu^2)^{1/2}(2L/R)^2(R/h) \geq 200$.

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